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Effect of insecticide on the population dynamics of brown planthoppers and *Cyrtorhinus lividipennis*: a modeling approach

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Abstract

In this paper, we investigate the effect of insecticide on the population dynamics of brown planthoppers (a major insect pest of rice) and *Cyrtorhinus lividipennis* (one of the natural enemies of brown planthoppers) by developing an impulsive mathematical model, and we analyze the model theoretically and numerically. The conditions on the parameters of the model for which the stability and permanence of the system can be ensured are derived. Computer simulations are also presented to confirm our theoretical results. The results show that with an appropriate amount and period between two consecutive applications of insecticide, the population of brown planthoppers could be maintained below a certain level, while *Cyrtorhinus lividipennis* could also survive.

Keywords: Mathematical model; Brown planthopper; *Cyrtorhinus lividipennis*; Insecticide

1 Introduction

The brown planthopper (BPH) is recognized as one of the major insect pests of rice. The well-known damage caused by the infestation of brown planthoppers is hopperburn in which the rice crop are wilting and drying completely [1]. The outbreak of BPH in Thailand during the dry season of the year 2010 caused approximately $52 million losses as reported by the Office of Agricultural Economics, the Ministry of Agriculture and Cooperatives of Thailand [2].

Biological control and insecticide have been used to control the outbreak of brown planthoppers in a paddy field. However, when insecticide is utilized not only brown planthoppers are eliminated but also its natural enemies in the paddy field such as *Cyrtorhinus lividipennis* [3, 4]. Even though insecticides have been widely used for controlling the pest, BPH has developed resistance to some major insecticides such as carbamates, organophosphates, neonicotinoids, phenylpyrazoles and pyrethroids [5, 6]. As it is quick and cost-effective against insects, chemical control is a popular choice in pest management. However, excessive and irrational use of chemical pesticides could lead to negative effects on the environment such as biodiversity's reduction and the decrease in population of natural enemies. Alternatively, biological control is a safe and an effective method. *Cyrtorhinus Lividipennis* is a major natural enemy of BPH. It preys mainly on eggs and nymphs of BPH [7, 8]. The predatory activity of *Cyrtorhinus lividipennis* against BPH has been investigated by many researchers and the study indicated that the *Cyrtorhinus lividipennis*'s preying on BPH's eggs was an important cause of the decrease in the BPH population [7, 9]. However, when the outbreak of BPH is severe, the use of *Cyrtorhinus lividipennis* alone might not be the most effective choice because the reproduction of *Cyrtorhinus lividipennis* is not rapid enough to control the outbreak.

In this paper, we investigate the effects of impulsive applications of insecticide on the population dynamics of brown planthoppers and *Cyrtorhinus lividipennis*.

2 An impulsive system

Let $B(t)$ represent the population density of brown planthoppers at time t and $C(t)$ represent the population density of *Cyrtorhinus lividipennis* at time t. The following impulsive system is proposed to investigate the population dynamics of brown planthoppers and *Cyrtorhinus lividipennis* when insecticide is utilized:

$$\left.\begin{aligned} \frac{dB}{dt} &= a_1B\left(1-\frac{B}{h_1}\right)-\frac{b_1BC}{1+h_2B}-d_1B \qquad &(1a)\\ \frac{dC}{dt} &= a_2C\left(1-\frac{C}{h_3}\right)+\frac{r_1b_1BC}{1+h_2B}-d_2C \qquad &(1b)\end{aligned}\right\} \quad t \neq kT,$$

with

$$\left.\begin{aligned} B\left(t^+\right) &= (1-\alpha)B(t) \qquad &(1c)\\ C\left(t^+\right) &= (1-\beta)C(t) \qquad &(1d)\end{aligned}\right\} \quad t = kT.$$

Here, $a_1, a_2, b_1, r_1, h_1, h_2, h_3, d_1$ and d_2 are assumed to be positive, T accounts for the period between two consecutive applications of insecticide, $k \in Z_+$, $Z_+ = \{1, 2, 3, \ldots\}$, α accounts for the negative effect of the insecticide on the population of brown planthoppers, $0 < \alpha < 1$, and β accounts for the negative effect of insecticide on the population of *Cyrtorhinus lividipennis*, $0 < \beta < 1$.

Equation (1a) describes the rate of change of the population of brown planthoppers. On the right-hand side, the first term represents the reproduction of brown planthoppers which is assumed to follow the logistic growth function, the second term represents the decrease in the population of brown planthoppers due to the predation by *Cyrtorhinus lividipennis* when the functional response is assumed to follow the Holling type II functional response and the last term represents the natural death rate of brown planthoppers.

Equation (1b) describes the rate of change of the population of *Cyrtorhinus lividipennis*. On the right-hand side, the first term represents the reproduction of *Cyrtorhinus lividipennis* which is assumed to follow the logistic growth function. Here, we assume that *Cyrtorhinus lividipennis* could feed on other insect pest in the paddy field apart from brown planthoppers and the population of *Cyrtorhinus lividipennis* could be high even as brown planthoppers are absent as reported in [10]. The second term represents the increase in the population of *Cyrtorhinus lividipennis* due to the predation on brown planthoppers and the last term represents the natural death rate of *Cyrtorhinus lividipennis*.

3 Theoretical results

Definition 1 Let the map defined by the right hand side of (1a)–(1d) be denoted by $f = (f_1, f_2)$ and let $G : R_+ \times R_+^2 \to R_+$, where $R_+ = [0, \infty), R_+^2 = \{X \in R^2 : X = (B, C), B \geq 0, C \geq 0\}$.

(a) G is said to belong to class G_0 if G is continuous in $(kT, (k+1)T] \times R_+^2 \to R_+$ and for each $X \in R_+^2, k \in Z_+$,

$$\lim_{(t,Y)\to(kT^+,X)} G(t, Y) = G\big(kT^+, X\big)$$

exists and G is locally Lipschitzian in X.

(b) Suppose $G \in G_0$. For $(t, X) \in (kT, (k+1)T] \times R_+^2$, the upper right derivative of $G(t, X)$ with respect to (1a)–(1d) is defined by

$$D^+G(t,X) = \limsup_{h\to 0^+} \frac{1}{h}\big[G\big(t+h, X + hf(t,X)\big) - G(t,X)\big].$$

The solution of (1a)–(1d), $X(t) = (B(t), C(t))$, is assumed to be a piecewise continuous function in what follows. This implies that $X(t) : R_+ \to R_+^2$, $X(t)$ is continuous on $(kT, (k+1)T]$, $k \in Z_+$ and $\lim_{t\to kT^+} X(t) = X(kT^+)$ exists. Therefore, the smoothness properties of f ensure the existence and uniqueness of solution to (1a)–(1d).

Consider (1a) and (1b) when $t \neq kT$. We can see that if $B(t) = 0$, $\frac{dB}{dt} = 0$ and if $C(t) = 0$, $\frac{dC}{dt} = 0$. In addition, $B(kT^+) = (1-\alpha)B(kT)$, $0 < \alpha < 1$, $C(kT^+) = (1-\beta)C(kT)$, $0 < \beta < 1$. Hence, the following lemma is obtained.

Lemma 1 *Let $X(t) = (B(t), C(t))$ be a solution of* (1a)–(1d). *Then $X(t) \geq 0$ for all $t \geq 0$ provided that $X(0^+) \geq 0$.*

Lemma 2 *Let $X(t) = (B(t), C(t))$ be a solution of* (1a)–(1d). *Then $B(t) \leq M^*$ and $C(t) \leq M^*$ for some constant $M^* > 0$ provided that*

$$d_2 > \frac{r_1 b_1}{h_2} \tag{2}$$

when t is sufficiently large.

Proof

Let $g(t) = B(t) + C(t)$, $M_1 = \frac{a_1 h_1}{4}$, $M_2 = \frac{a_2 h_3}{4}$, $M_3 = \frac{r_1 b_1}{h_2}$ and $c^* = \min\{d_1, d_2 - M_3\}$.

For $t \neq kT$, we can see that

$$\begin{aligned}
&D^+g + c^*g \\
&\quad= \frac{dB}{dt} + \frac{dC}{dt} + c^*B + c^*C \\
&\quad= a_1 B\left(1 - \frac{B}{h_1}\right) - \frac{b_1 BC}{1 + h_2 B} - d_1 B + a_2 C\left(1 - \frac{C}{h_3}\right) + \frac{r_1 b_1 BC}{1 + h_2 B} - d_2 C + c^*B + c^*C \\
&\quad\leq \big(c^* - d_1\big)B + \big(c^* - d_2 + M_3\big)C + M_1 + M_2 \\
&\quad\leq M_1 + M_2 \equiv M_0.
\end{aligned}$$

Hence, $D^+g \leq -c^*g + M_0$.

For $t = kT$,

$$g(kT^+) = B(kT^+) + C(kT^+) = (1-\alpha)B(kT) + (1-\beta)C(kT)$$
$$= B(kT) + C(kT) - \alpha B(kT) - \beta C(kT)$$
$$\le g(kT).$$

For $t \in (kT,(k+1)T]$, Lemma 2.2 of [11] implies that

$$g(t) \le g(0)e^{-c^*t} + \int_0^t M_0 e^{-c^*(t-x)}\,dx$$
$$\le g(0)e^{-c^*t} + M_0\left(\frac{1}{c^*} - \frac{e^{-c^*t}}{c^*}\right)$$
$$< \frac{M_0}{c^*} \equiv M^* \quad \text{as } t \to \infty.$$

Since $g(t) = B(t) + C(t)$ and $g(t) < M^*$, this means that $B(t) \le M^*$ and $C(t) \le M^*$ when t is large enough and $M^* > 0$. □

Next, let us investigate the reduced system of (1a)–(1d) when the brown planthopper is absent ($B = 0$):

$$\frac{dC}{dt} = sC - rC^2, \quad t \ne kT, \tag{3}$$
$$C(kT^+) = (1-\beta)C(kT), \quad t = kT, \tag{4}$$
$$C(0^+) = C_0, \tag{5}$$

where $r \equiv \frac{a_2}{h_3} > 0$ and $s \equiv a_2 - d_2$. Suppose that $s > 0$ that is

$$a_2 > d_2. \tag{6}$$

We can see that the solution of (3) is

$$C(t) = \frac{s}{r + c_1 s e^{-st}},$$

where c_1 is an arbitrary constant.

Since $C(t)$ is an increasing function for $s > 0$ and (4), the system (3)–(5) has a periodic solution

$$\tilde{C}(t) = \frac{s(1-\beta-e^{-sT})}{\beta r e^{-s(t-kT)} + r(1-\beta-e^{-sT})}), \quad t \in (kT,(k+1)T], \tag{7}$$

with $\tilde{C}(0^+) = \frac{s(1-\beta-e^{-sT})}{r(1-e^{-sT})} > 0$ and (6) holding.

Therefore,

$$C(t) = \frac{C_0 s(1-\beta-e^{-sT})\tilde{C}(t)e^{st}}{[s(1-\beta-e^{-sT}) - C_0 r(1-e^{-sT})]\tilde{C}(t) + C_0 s(1-\beta-e^{-sT})e^{st}}, \quad t \in (kT,(k+1)T],$$

is the positive solution of (3)–(5).

Lemma 3 *The system* (3)–(5) *has a positive periodic solution* $\tilde{C}(t)$, *and* $C(t) \to \tilde{C}(t)$ *as* $t \to \infty$ *for every solution* $C(t)$ *of* (3)–(5).

Therefore,

$$\left(0, \tilde{C}(t)\right) = \left(0, \frac{s(1-\beta-e^{-sT})}{\beta r e^{-s(t-kT)} + r(1-\beta-e^{-sT})}\right)$$

is a periodic solution of the system (1a)–(1d) at the absence of brown planthoppers for $t \in (kT, (k+1)T]$ and $\tilde{C}(kT^+) = \tilde{C}(0^+) = \frac{s(1-\beta-e^{-sT})}{r(1-e^{-sT})}, k \in Z_+$.

Theorem 1 *Suppose that*

$$T_2 < T < T_1, \tag{8}$$

$$a_1 > d_1 + \frac{b_1 s}{r}, \tag{9}$$

and

$$\ln\left(\frac{1}{1-\alpha}\right) > \frac{b_1}{r} \ln\left(\frac{1}{1-\beta}\right). \tag{10}$$

Then the solution $(0, \tilde{C}(t))$ *of* (1a)–(1d) *is locally asymptotically stable where* $T_1 = \frac{1}{(a_1-d_1-\frac{b_1 s}{r})}[\ln(\frac{1}{1-\alpha}) - \frac{b_1}{r}\ln(\frac{1}{1-\beta})]$ *and* $T_2 = \frac{1}{s}\ln(\frac{1}{1-\beta})$.

Proof Let us consider a small perturbation

$$B(t) = v_1(t),$$

$$C(t) = \tilde{C}(t) + v_2(t),$$

from the point $(0, \tilde{C}(t))$. Then

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}, \quad 0 < t < T,$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a_1 - d_1 - b_1\tilde{C}(t) & 0 \\ * & s - 2r\tilde{C}(t) \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$, the identity matrix. Hence, the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp \int_0^t (a_1 - d_1 - b_1\tilde{C}(x))\,dx & 0 \\ ** & \exp \int_0^t (s - 2r\tilde{C}(x))\,dx \end{pmatrix}.$$

Note that the terms (*) and (**) are not involved in the further calculations and hence it is not necessary to find (*) and (**).

Linearization of (1c)–(1d) yields

$$\begin{pmatrix} v_1(kT^+) \\ v_2(kT^+) \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{pmatrix} \begin{pmatrix} v_1(kT) \\ v_2(kT) \end{pmatrix}.$$

Consider

$$A = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{pmatrix} \Phi(T).$$

The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= (1-\alpha)\exp\int_0^T \bigl(a_1 - d_1 - b_1\tilde{C}(x)\bigr)\,dx \\ &= (1-\alpha)\exp\biggl((a_1-d_1)T - \frac{b_1}{r}\bigl(\ln(1-\beta)+sT\bigr)\biggr), \\ \lambda_2 &= (1-\beta)\exp\int_0^T \bigl(s - 2r\tilde{C}(x)\bigr)\,dx = (1-\beta)\exp\bigl(-sT - 2\ln(1-\beta)\bigr). \end{aligned}$$

Since $0 < \alpha < 1$, $0 < \beta < 1$, (8)–(10) hold, and then

$$\frac{1}{s}\ln\biggl(\frac{1}{1-\beta}\biggr) < T < \frac{1}{(a_1 - d_1 - \frac{b_1 s}{r})}\biggl[\ln\biggl(\frac{1}{1-\alpha}\biggr) - \frac{b_1}{r}\ln\biggl(\frac{1}{1-\beta}\biggr)\biggr].$$

Hence,

$$|\lambda_1| = (1-\alpha)\exp\int_0^T \bigl(a_1 - d_1 - b_1\tilde{C}(x)\bigr)\,dx < 1$$

and

$$|\lambda_2| = (1-\beta)\exp\bigl(-sT - 2\ln(1-\beta)\bigr) < 1.$$

All conditions of Floquet theory are now satisfied and, hence, we can conclude that the solution $(0, \tilde{C}(t))$ of (1a)–(1d) is locally stable, which completes the proof. □

System Permanence

Definition 2 If there exist constants $n, m > 0$ and a finite time t_0 such that, for all solution with all initial values $B(0^+) > 0$ and $C(0^+) > 0$,

$$\begin{aligned} &n \le B(t) \le m, \\ &n \le C(t) \le m, \end{aligned}$$

for all $t > t_0$, the system (1a)–(1d) is said to be permanent where we note that t_0 may depend on the initial values whereas n, m are independent of the initial values.

Theorem 2 *The system* (1a)–(1d) *is permanent if*

$$T > T^* \tag{11}$$

and

$$r > s + M_3 \tag{12}$$

provided (2), (6), (9) *and* (10) *hold where*

$$T^* \equiv \frac{1}{(a_1 - d_1 - \frac{b_1(s+M_3)}{r})}\left[\ln\left(\frac{1}{1-\alpha}\right) - \frac{b_1}{r}\ln\left(\frac{1}{1-\beta}\right)\right].$$

Proof Let $X(t) = (B(t), C(t))$ be a solution of the system (1a)–(1d) with $B(0^+) > 0$ and $C(0^+) > 0$. For sufficiently large t, Lemma 2 implies that a constant $m > 0$ exists so that $B(t) \leq m$ and $C(t) \leq m$.

Since $\frac{r_1 b_1 BC}{1+k_2 B} \geq 0$, (1b) implies that

$$\begin{aligned} &\frac{dC}{dt} \geq sy - ry^2, \quad t \neq kT, \\ &C(kT^+) = (1-\beta)C(kT), \quad t = kT, \end{aligned}$$

and for sufficiently large t, we have

$$C(t) > \tilde{C}(t) - \varepsilon$$

for some $\varepsilon > 0$.

Hence, for sufficiently large t, we obtain

$$C(t) > \frac{s(1-\beta-e^{-sT})}{r(1-e^{-sT})} - \varepsilon \equiv n_1.$$

Next, we have to show that there exists a constant $n_2 > 0$ such that $B(t) > n_2$. For some $n_3 > 0$, let

$$\hat{M}_1 = a_1\left(1 - \frac{n_3}{h_1}\right) - d_1.$$

Step 1. In order to prove by contradiction that there exists t_1 such that $B(t_1) \geq n_3$, we assume that $B(t) < n_3$ for all positive t.

Equations (1b) and (1d) imply that

$$\begin{aligned} \frac{dC}{dt} &= a_2 C\left(1 - \frac{C}{h_3}\right) + \frac{r_1 b_1 BC}{1 + h_2 B} - d_2 C, \quad t \neq kT \\ &\leq a_2 C\left(1 - \frac{C}{h_3}\right) + M_3 C - d_2 C \\ &= (s + M_3)C - rC^2, \end{aligned}$$

$$C(t^+) = (1-\beta)C(t), \quad t = kT.$$

Consider the comparison system

$$\frac{dZ}{dt} = (s+M_3)Z - rZ^2, \quad t \neq kT, \tag{13}$$

$$Z(t^+) = (1-\beta)Z(t), \quad t = kT, \tag{14}$$

and

$$Z(0^+) = C(0^+). \tag{15}$$

Hence,

$$\frac{1}{\tilde{Z}(t)} = \frac{\beta r e^{-(s+M_3)(t-kT)}}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} + \frac{r}{(s+M_3)}, \quad t \in (kT,(k+1)T], \tag{16}$$

is a periodic solution of (13)–(15) with $\frac{1}{Z(0^+)} = \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} + \frac{r}{(s+M_3)} > 0$. The positive solution of (13)–(15) is

$$\frac{1}{Z(t)} = \left(\frac{1}{Z(0^+)} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right)e^{-(s+M_3)t} + \frac{1}{\tilde{Z}(t)}, \tag{17}$$

$t \in (kT,(k+1)T]$ and as $t \to \infty$

$$\frac{1}{Z(t)} \to \frac{1}{\tilde{Z}(t)} = \frac{\beta r e^{-(s+M_3)(t-kT)}}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} + \frac{r}{(s+M_3)}.$$

Hence, we can conclude that $C(t) \leq Z(t)$ by the comparison theorem [12].

Now, we consider (1a)

$$\begin{aligned}\frac{dB}{dt} &= a_1 B\left(1-\frac{B}{h_1}\right) - \frac{b_1 BC}{1+h_2 B} - d_1 B \\ &\geq \left(a_1\left(1-\frac{n_3}{h_1}\right) - b_1 C - d_1\right)B \\ &= (\hat{M}_1 - b_1 C)B.\end{aligned}$$

Since $C(t) \leq Z(t)$, there is a $T_1 > 0$ such that

$$C(t) \leq Z(t) < \tilde{Z}(t) + \varepsilon_1, \quad t \neq kT, t \geq T_1,$$

for a sufficiently small $\varepsilon_1 > 0$.

Therefore,

$$\frac{dB}{dt} > \left(\hat{M}_1 - b_1\left(\tilde{Z}(t) + \varepsilon_1\right)\right)B, \quad t \neq kT, t \geq T_1, \tag{18}$$

and

$$B(t^+) = (1-\alpha)B(t), \quad t = kT, t \geq T_1. \tag{19}$$

Letting $K \in Z_+$ and $KT \geq T_1$, and integrating over $(kT,(k+1)T], k \geq K$, we get

$$\begin{aligned} B\big((k+1)T\big) &\geq B(kT)(1-\alpha)\exp\left(\int_{kT}^{(k+1)T}\big(\hat{M}_1 - b_1\big(\tilde{Z}(t)+\varepsilon_1\big)\big)\,dt\right) \\ &= B(kT)(1-\alpha)\exp\left(\left(\hat{M}_1 - b_1\varepsilon_1 - \frac{b_1(s+M_3)}{r}\right)T + \frac{b_1}{r}\ln\left(\frac{1}{1-\beta}\right)\right) \\ &= B(kT)\gamma, \end{aligned}$$

where $\gamma \equiv (1-\alpha)\exp((\hat{M}_1 - b_1\varepsilon_1 - \frac{b_1(s+M_3)}{r})T + \frac{b_1}{r}\ln(\frac{1}{1-\beta}))$.

Consider

$$\ln\gamma = \ln(1-\alpha) + \left(\hat{M}_1 - b_1\varepsilon_1 - \frac{b_1(s+M_3)}{r}\right)T + \frac{b_1}{r}\ln\left(\frac{1}{1-\beta}\right).$$

For sufficiently small $\varepsilon_1 > 0$,

$$\ln\gamma \approx \left(\hat{M}_1 - \frac{b_1(s+M_3)}{r}\right)T + \frac{b_1}{r}\ln\left(\frac{1}{1-\beta}\right) - \ln\left(\frac{1}{1-\alpha}\right).$$

Since $\hat{M}_1 < a_1 - d_1$ and (9) hold, we can choose a small $n_3 > 0$ such that $\ln\gamma > 0$ and, hence,

$$\gamma \equiv (1-\alpha)\exp\left(\left(\hat{M}_1 - b_1\varepsilon_1 - \frac{b_1(s+M_3)}{r}\right)T + \frac{b_1}{r}\ln\left(\frac{1}{1-\beta}\right)\right) > 1. \tag{20}$$

Then $B((k+i)T) \geq B(kT)\gamma^i \to \infty$ as $i \to \infty$. It is in contradiction to the boundedness of $B(t)$. Hence, there is $t_1 > 0$ such that $B(t_1) \geq n_3$.

Step 2. If $B(t) \geq n_3$ for all $t > t_1$, then the proof is complete. Otherwise, $B(t) < n_3$ for some $t > t_1$. Let $t^* = \inf_{t>t_1}\{B(t) < n_3\}$.

Case 1: $t^* = k_1 T$ for some $k_1 \in Z_+$. That is, for $t \in (t_1, t^*]$, $B(t) \geq n_3$ and the continuity of $B(t)$ implies that $B(t^*) = n_3$.

Since there are $m > 0$ and $n_1 > 0$ such that, for sufficiently large t, $B(t) < m$ and $n_1 < C(t) < m$, $m' > 0$ and $n_1' > 0$ are chosen so that

$$B(t) < m' \quad \text{and} \quad n_1' < C(t) < m'$$

and

$$\hat{M}_1 < b_1 m' \tag{21}$$

such that

$$\left|\frac{1}{C(t^{*+})} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right| - \beta < \frac{1}{n'}. \tag{22}$$

Then choose $k_2, k_3 \in Z_+$ such that

$$k_2 T > \frac{1}{(s+M_3)}\ln\left(\frac{\frac{1}{n'}+\beta}{\varepsilon_1}\right) \tag{23}$$

and

$$(1-\alpha)^{k_2}\exp\bigl((k_2+1)\gamma_1 T\bigr)\gamma^{k_3} > 1, \tag{24}$$

where

$$\gamma_1 \equiv \hat{M}_1 - b_1 m' < 0.$$

Let $T' = k_2T + k_3T$. We claim that there is $t_2 \in (t^*, t^* + T']$ such that $B(t_2) > n_3$. Otherwise, considering (17) with $\frac{1}{Z(t^{*+})} = \frac{1}{C(t^{*+})}$, we have

$$\frac{1}{Z(t)} = \left(\frac{1}{Z(t^{*+})} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right)e^{-(s+M_3)(t-t^*)} + \frac{1}{\tilde{Z}(t)}$$

for $t \in (kT,(k+1)T]$ and $k_1 \le k \le k_1 + k_2 + k_3$.

For $k_2T \le t - t^* \le T'$, we have

$$\begin{aligned}
\left|\frac{1}{Z(t)} - \frac{1}{\tilde{Z}(t)}\right| &= \left|\frac{1}{Z(t^{*+})} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right|e^{-(s+M_3)(t-t^*)} \\
&= \left|\frac{1}{C(t^{*+})} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right|e^{-(s+M_3)(t-t^*)} \\
&< \left(\frac{1}{n'} + \beta\right)e^{-(s+M_3)(t-t^*)} \\
&< \left(\frac{1}{n'} + \beta\right)e^{-(s+M_3)k_2T} \\
&< \varepsilon_1.
\end{aligned}$$

Since (12), we have

$$\bigl|Z(t) - \tilde{Z}(t)\bigr| < \frac{|Z(t)-\tilde{Z}(t)|}{|Z(t)\tilde{Z}(t)|} = \left|\frac{1}{Z(t)} - \frac{1}{\tilde{Z}(t)}\right| < \varepsilon_1.$$

Then

$$C(t) \le Z(t) < \tilde{Z}(t) + \varepsilon_1.$$

Similar to *Step* 1, we have

$$\begin{aligned}
B\bigl(t^* + T'\bigr) &= B(k_1T + k_2T + k_3T) \\
&\ge B\bigl(t^* + k_2T\bigr)\gamma^{k_3}.
\end{aligned}$$

From (1a), we have

$$\begin{aligned}
\frac{dB}{dt} &= a_1B\left(1 - \frac{B}{h_1}\right) - \frac{b_1BC}{1+h_2B} - d_1B, \quad t \ne kT \\
&\ge \bigl(\hat{M}_1 - b_1m'\bigr)B = \gamma_1 B,
\end{aligned} \tag{25}$$

$$B(t^+) = (1-\alpha)B(t), \quad t = kT,$$

and then integrating over $[t^*, t^* + k_2T]$, we obtain

$$\begin{aligned} B(t^* + k_2T) &\geq B(t^*)(1-\alpha)^{k_2} \exp\left(\int_{k_1T}^{k_1T+k_2T} \gamma_1 \, dt\right) \\ &\geq n_3(1-\alpha)^{k_2} \exp(k_2\gamma_1 T) \\ &\geq n_3(1-\alpha)^{k_2} \exp\big((k_2+1)\gamma_1 T\big) \end{aligned}$$

and hence

$$\begin{aligned} B(t^* + T') &\geq B(t^* + k_2T)\gamma^{k_3} \\ &\geq n_3(1-\alpha)^{k_2} \exp\big((k_2+1)\gamma_1 T\big)\gamma^{k_3} \\ &> n_3. \end{aligned}$$

Hence, the definition of n_3 is contradicted. Therefore, there exists $t_2 \in (t^*, t^* + T']$ such that $B(t_2) > n_3$.

Now, let $\tilde{t} = \inf_{t>t^*}\{B(t) > n_3\}$. Then $B(t) < n_3$ for $t \in (t^*, \tilde{t})$, and the continuity of $B(t)$ implies that $B(\tilde{t}) = n_3$. Next, we choose $q \in Z_+$ such that $q \leq k_2 + k_3$ and $t^* + qT \geq \tilde{t}$, and suppose $t \in (t^* + (q-1)T, t^* + qT]$. From (25), we have

$$\begin{aligned} B(t) &\geq B(t^{*+})(1-\alpha)^{q-1} \exp\big((q-1)\gamma_1 T\big) \exp\Big(\gamma_1\big(t - \big(t^* + (q-1)T\big)\big)\Big) \\ &= B(t^*)(1-\alpha)^{q} \exp\big((q-1)\gamma_1 T\big) \exp\Big(\gamma_1\big(t - \big(t^* + (q-1)T\big)\big)\Big) \\ &= n_3(1-\alpha)^{q} \exp\big(\gamma_1(t - t^*)\big) \\ &\geq n_3(1-\alpha)^{k_2+k_3} \exp(\gamma_1 qT) \\ &\geq n_3(1-\alpha)^{k_2+k_3} \exp\big((k_2+k_3)\gamma_1 T\big). \end{aligned}$$

We used $\gamma_1 < 0$ and $q \leq k_2 + k_3$.

Letting

$$\bar{n}_2 = n_3(1-\alpha)^{k_2+k_3} \exp\big((k_2+k_3)\gamma_1 T\big),$$

we can see that $B(t) \geq \bar{n}_2$ for $t \in (t^*, \tilde{t})$. By using $\tilde{t}$ instead of t^* and continuing in the same way, we then obtain $B(t) \geq \bar{n}_2$ for all t large enough.

Case 2: $t^* \neq kT$ for all $k \in Z_+$. This implies that $B(t) \geq n_3$ for $t \in [t_1, t^*)$ and $B(t^*) = n_3$. For some $k_1' \in Z_+$, suppose that $t^* \in (k_1'T, (k_1'+1)T)$.

Case 2.1: $B(t) \leq n_3$ for all $t \in (t^*, (k_1'+1)T]$. We claim that there is $t_2' \in [(k_1'+1)T, (k_1'+1)T + T']$ such that $x(t_2') > n_3$. Otherwise, considering (17) with $\frac{1}{Z((k_1'+1)T^+)} = \frac{1}{C((k_1'+1)T^+)}$. For $t \in (kT, (k+1)T]$, $k_1' + 1 \leq k \leq k_1' + 1 + k_2 + k_3$, we obtain

$$\begin{aligned} \frac{1}{Z(t)} = {}& \left(\frac{1}{Z((k_1'+1)T^+)} - \frac{\beta r}{(s+M_3)(1-\beta-e^{-(s+M_3)T})} - \frac{r}{(s+M_3)}\right) \\ &\times e^{-(s+M_3)(t-(k_1'+1)T)} + \frac{1}{\tilde{Z}(t)}. \end{aligned}$$

For $k_2T \leq t - t^*$, as in Case 1, we obtain

$$\left|Z(t) - \tilde{Z}(t)\right| < \varepsilon_1.$$

Then

$$C(t) \leq Z(t) < \tilde{Z}(t) + \varepsilon_1.$$

Since $k_2T \leq (k_1' + 1 + k_2)T - t^*$, we have

$$\begin{aligned} B\big((k_1' + 1 + k_2)T\big) &\geq B\big(t^*\big)(1-\alpha)^{k_2} \exp\big(\gamma_1\big((k_1' + 1 + k_2)T - t^*\big)\big) \\ &\geq n_3(1-\alpha)^{k_2} \exp\big(\gamma_1\big((k_1' + 1 + k_2)T - k_1'T\big)\big) \\ &\geq n_3(1-\alpha)^{k_2} \exp\big((k_2+1)\gamma_1 T\big). \end{aligned}$$

Then

$$\begin{aligned} B\big((k_1' + 1 + k_2 + k_3)T\big) &\geq B\big((k_1' + 1 + k_2)T\big)\gamma^{k_3} \\ &\geq n_3(1-\alpha)^{k_2} \exp\big((k_2+1)\gamma_1 T\big)\gamma^{k_3} \\ &> n_3. \end{aligned}$$

The definition of n_3 is contradicted and, hence, we can conclude that there is $t_2' \in [(k_1' + 1)T, (k_1' + 1)T + T']$ such that $B(t_2') > n_3$.

Now, let $\bar{t} = \inf_{t>t^*}\{B(t) > n_3\}$. Then $B(t) \leq n_3$ for $t \in [t^*, \bar{t})$, and $B(\bar{t}) = n_3$. We choose $q' \in Z_+$ such that $q' \leq k_2 + k_3 + 1$ and suppose $t \in (k_1'T + (q'-1)T, k_1'T + q'T]$. From (25), we have

$$\begin{aligned} B(t) &\geq B\big(\big(k_1'T + \big(q'-1\big)T\big)^+\big) \exp\big(\gamma_1\big(t - \big(k_1'T + \big(q'-1\big)T\big)\big)\big) \\ &= B\big(k_1'T + \big(q'-1\big)T\big)(1-\alpha) \exp\big(\gamma_1\big(t - \big(k_1'T + \big(q'-1\big)T\big)\big)\big) \\ &\geq B\big(t^*\big)(1-\alpha)^{q'-1} \exp\big(\gamma_1\big(t - t^*\big)\big) \\ &\geq n_3(1-\alpha)^{q'-1} \exp\big(\gamma_1\big(t - t^*\big)\big). \end{aligned}$$

We used $\gamma_1 < 0$ and $t - t^* \leq q'T$. Hence,

$$B(t) \geq n_3(1-\alpha)^{k_2+k_3} \exp\big((k_2 + k_3 + 1)\gamma_1 T\big).$$

Letting

$$n_2 = n_3(1-\alpha)^{k_2+k_3} \exp\big((k_2 + k_3 + 1)\gamma_1 T\big),$$

then, for $t \in (t^*, \bar{t})$, we obtain $B(t) \geq n_2$. By using $\bar{t}$ instead of t^* and continue with the same way, we shall obtain $B(t) \geq n_2$ for all t large enough.

Case 2.2: There is a $t'' \in (t^*, (k_1' + 1)T]$ such that $B(t'') > n_3$. Let $\underline{t} = \inf_{t>t^*}\{B(t) > n_3\}$. Hence, $B(t) < n_3$ for $t \in [t^*, \underline{t})$, and $B(\underline{t}) = n_3$. For $t \in [t^*, \underline{t})$, (25) holds, we have

$$\begin{aligned} B(t) &\geq B(t^*) \exp\left(\int_{t^*}^{t} \gamma_1 \, dt\right) \\ &= n_3 \exp(\gamma_1 (t - t^*)) \\ &\geq n_3 \exp(\gamma_1 T) \\ &> n_2 \end{aligned}$$

since $t < k_1' T + T < t^* + T$.

Since $B(\underline{t}) \geq n_3$, we can continue in the same way for $t > \underline{t}$. Since $n_2 < \bar{n}_2 < n_3$, we have $B(t) \geq n_2$ for $t \geq t_1$. The proof is complete. □

Existence of the positive periodic solution

Let us investigate the possibility of positive periodic solution to the system (1a)–(1d) near $(0, \tilde{C}(t))$ by interchanging the state variables and consider the following system instead:

$$\frac{dB}{dt} = a_2 B\left(1 - \frac{B}{h_3}\right) + \frac{r_1 b_1 BC}{1 + h_2 C} - d_2 B, \quad t \neq kT, \tag{26}$$

$$\frac{dC}{dt} = a_1 C\left(1 - \frac{C}{h_1}\right) - \frac{b_1 BC}{1 + h_2 C} - d_1 C, \quad t \neq kT, \tag{27}$$

with

$$\Delta B(t) = -\beta B(t), \quad t = kT, \tag{28}$$

$$\Delta C(t) = -\alpha C(t), \quad t = kT. \tag{29}$$

Let

$$f_1(B, C) = a_2 B\left(1 - \frac{B}{h_3}\right) + \frac{r_1 b_1 BC}{1 + h_2 C} - d_2 B, \qquad f_2(B, C) = a_1 C\left(1 - \frac{C}{h_1}\right) - \frac{b_1 BC}{1 + h_2 C} - d_1 C.$$

According to Lakmeche and Arini [13],

$$\begin{aligned} &\Theta_1(B, C) = (1 - \beta)B, \qquad \Theta_2(B, C) = (1 - \alpha)C, \qquad \varsigma(t) = \left(\tilde{C}(t), 0\right)^T, \\ &X_0 = \left(\tilde{C}(\tau_0), 0\right)^T, \qquad \tau_0 = T_1, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Phi_1(\tau_0, X_0)}{\partial \tau} &= \frac{\partial \tilde{C}(\tau_0, X_0)}{\partial t} = \frac{\beta r \exp(-s\tau_0) \tilde{C}^2(\tau_0, X_0)}{1 - \beta - \exp(-s\tau_0)} > 0, \\ \frac{\partial \Phi_1(\tau_0, X_0)}{\partial B} &= \exp\left(\int_0^{\tau_0} \frac{\partial f_1(\varsigma(u))}{\partial B} \, du\right) > \frac{1}{1 - \beta} > 0, \\ \frac{\partial \Phi_1(\tau_0, X_0)}{\partial C} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\tau_0} \exp\left(\int_\upsilon^{\tau_0} \frac{\partial f_1(\varsigma(u))}{\partial B}\,du\right) \frac{\partial f_1(\varsigma(\upsilon))}{\partial C} \exp\left(\int_0^{\upsilon} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) d\upsilon \\
&= \int_0^{\tau_0} \exp\left(\int_\upsilon^{\tau_0} \bigl(s - 2r\tilde{C}(u)\bigr)\,du\right) \bigl(r_1 b_1 \tilde{C}(\upsilon)\bigr) \\
&\quad \times \exp\left(\int_0^{\upsilon} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right) d\upsilon,
\end{aligned}
$$

$$
\frac{\partial \Phi_2(\tau_0, X_0)}{\partial C} = \exp\left(\int_0^{\tau_0} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) = \exp\left(\int_0^{\tau_0} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right),
$$

$$
\begin{aligned}
&\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial B \partial C} \\
&= \int_0^{\tau_0} \exp\left(\int_\upsilon^{\tau_0} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) \frac{\partial^2 f_2(\varsigma(\upsilon))}{\partial B \partial C} \exp\left(\int_0^{\upsilon} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) d\upsilon \\
&= \frac{-b_1 \tau_0}{1-\alpha} < 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial C^2} \\
&= \int_0^{\tau_0} \exp\left(\int_\upsilon^{\tau_0} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) \frac{\partial^2 f_2(\varsigma(\upsilon))}{\partial C^2} \exp\left(\int_0^{\upsilon} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) d\upsilon \\
&\quad + \int_0^{\tau_0} \left[\exp\left(\int_\upsilon^{\tau_0} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) \frac{\partial^2 f_2(\varsigma(\upsilon))}{\partial B \partial C}\right] \\
&\quad \times \left[\int_0^{\upsilon} \exp\left(\int_\theta^{\upsilon} \frac{\partial f_1(\varsigma(u))}{\partial B}\,du\right) \frac{\partial f_1(\varsigma(\theta))}{\partial C} \exp\left(\int_0^{\theta} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) d\theta\right] d\upsilon \\
&= \int_0^{\tau_0} \left(-\frac{2a_1}{h_1} + 2h_1\right) \exp\left(\int_0^{\tau_0} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right) d\upsilon \\
&\quad - b_1 \int_0^{\tau_0} \left[\exp\left(\int_\upsilon^{\tau_0} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right)\right] \\
&\quad \times \left[\int_0^{\upsilon} \exp\left(\int_\theta^{\upsilon} \bigl(s - 2r\tilde{C}(u)\bigr)\,du\right) \bigl(r_1 b_1 \tilde{C}(\theta)\bigr)\right. \\
&\quad \left. \times \exp\left(\int_0^{\theta} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right) d\theta\right] d\upsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial C \partial \tau} \\
&= \frac{\partial f_2(\varsigma(\tau_0))}{\partial C} \exp\left(\int_0^{\tau_0} \frac{\partial f_2(\varsigma(u))}{\partial C}\,du\right) \\
&= \bigl(a_1 - d_1 - b_1\tilde{C}(\tau_0)\bigr) \exp\left(\int_0^{\tau_0} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right) \\
&= \frac{1}{1-\alpha}\left(a_1 - d_1 - \frac{b_1 s(1 - \beta - \exp(-s\tau_0))}{\beta r \exp(-s\tau_0) + r(1 - \beta - \exp(-s\tau_0))}\right).
\end{aligned}
$$

Now, we can compute

$$
d_0' = 1 - \left(\frac{\partial \Theta_2}{\partial C} \frac{\partial \Phi_2}{\partial C}\right)_{(\tau_0, X_0)} = 1 - (1-\alpha)\exp\left(\int_0^{\tau_0} \bigl(a_1 - d_1 - b_1\tilde{C}(u)\bigr)\,du\right),
$$

where τ_0 is the root of $d_0' = 0$. Note that $d_0' > 0$ if $T < T_1$ and $d_0' < 0$ if $T > T_1$.

$$a_0' = 1 - \left(\frac{\partial\Theta_1}{\partial B}\frac{\partial\Phi_1}{\partial B}\right)_{(\tau_0,X_0)} = 1 - (1-\beta)\exp\left(\int_0^{\tau_0}\left(s - 2r\tilde{C}(u)\right)du\right).$$

Note that $a_0' > 0$ if $T > T_1 > T_2$,

$$\begin{aligned}
b_0' &= -\left(\frac{\partial\Theta_1}{\partial B}\frac{\partial\Phi_1}{\partial C} + \frac{\partial\Theta_1}{\partial C}\frac{\partial\Phi_2}{\partial C}\right)_{(\tau_0,X_0)} \\
&= -\frac{\partial\Theta_1}{\partial B}\frac{\partial\Phi_1(\tau_0,X_0)}{\partial C} \\
&= -(1-\beta)\int_0^{\tau_0}\exp\left(\int_\upsilon^{\tau_0}\left(s - 2r\tilde{C}(u)\right)du\right)\left(r_1 b_1\tilde{C}(\upsilon)\right) \\
&\quad\times\exp\left(\int_0^\upsilon\left(a_1 - d_1 - b_1\tilde{C}(u)\right)du\right)d\upsilon < 0, \\
P^* &= -a_1 + d_1 + b_1 s\left(\frac{1-\beta-\exp(-s\tau_0)}{\beta\exp(-s\tau_0) + (1-\beta-\exp(-s\tau_0))}\right) \\
&\quad\times\left(\frac{1}{r} + \frac{\tau_0(1-\beta)\beta s(1-\beta-\exp(-s\tau_0))}{(1+\beta+\beta\exp(s\tau_0))(\beta\exp(-s\tau_0)+1-\beta-\exp(-s\tau_0))}\right), \\
Q^* &= 2(1-\alpha)\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2}{\partial B\partial C} - (1-\alpha)\frac{\partial^2\Phi_2}{\partial C^2}.
\end{aligned}$$

Note that $P^* < 0$ and $Q^* > 0$ if

$$h_1{}^2 < a_1 \tag{30}$$

and

$$s^2 + s - 1 < 0. \tag{31}$$

Thus, $P^*Q^* < 0$, and by Lakmeche and Arini [13], the following result is obtained.

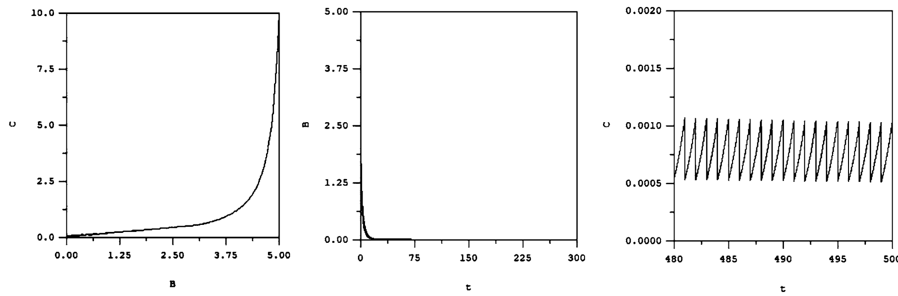
Theorem 3 *The system* (25)–(28) *has a positive periodic solution which is supercritical provided* (2), (6), (9), (10), (12), (30), (31) *hold, and* $T > T_1 > T_2$.

4 Numerical simulations

Figure 1 shows a simulation result of the system of Eqs. (1a)–(1d) with the parametric values $a_1 = 0.5, a_2 = 0.8, b_1 = 0.5$, $r_1 = 0.9$, $d_1 = 0.01, d_2 = 0.1, h_1 = 2, h_2 = 5, h_3 = 0.3$, $\alpha = 0.5, \beta = 0.5, T = 1$, $B(0) = 5$, and $C(0) = 10$ in which all the conditions in Theorem 1 are satisfied. The solution trajectory tends to a limit cycle as predicted in Theorem 1.

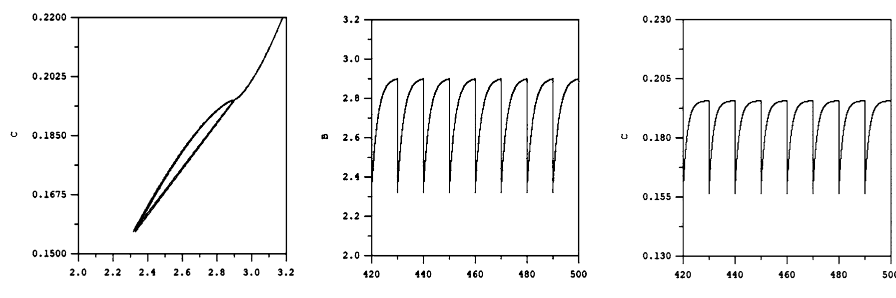
Figure 2 shows a simulation result of the system of equations (1a)–(1d) with the parametric values $a_1 = 0.5, a_2 = 0.7, b_1 = 0.5, r_1 = 0.9, d_1 = 0.01$, $d_2 = 0.1$, $h_1 = 3$, $h_2 = 5$, $h_3 = 0.2$, $\alpha = 0.2$, $\beta = 0.2$, $T = 10$, $B(0) = 5$, and $C(0) = 5$ in which all the conditions in Theorem 2 are satisfied. The solution of the system shows permanence as predicted in Theorem 2.

Figure 3 shows a simulation result of the system of equations (1a)–(1d) with the parametric values $a_1 = 0.5, a_2 = 0.4, b_1 = 0.5, r_1 = 0.9, d_1 = 0.01, d_2 = 0.2, h_1 = 0.3, h_2 = 2$, $h_3 = 0.5$, $\alpha = 0.5$, $\beta = 0.5, T = 5, B(0) = 5$, and $C(0) = 5$ in which all the conditions in Theorem 3 are satisfied. The solution of the system is positive periodic as predicted in Theorem 3.



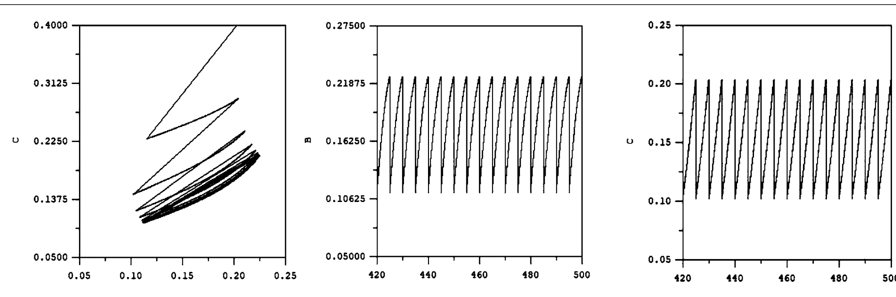
(a) (b) (c)

Figure 1 A computer simulation of Eqs. (1a)–(1d). The solution trajectory tends toward the oscillatory solution $(0, \tilde{C}(t))$ as time progresses. Here, $a_1 = 0.5, a_2 = 0.8, b_1 = 0.5, r_1 = 0.9,$ $d_1 = 0.01, d_2 = 0.1, h_1 = 2, h_2 = 5, h_3 = 0.3, \alpha = 0.5, \beta = 0.5, T = 1, B(0) = 5,$ and $C(0) = 10$. Here, all conditions in Theorem 1 are satisfied. (**a**) The solution trajectory projected on the (B, C)-plane. (**b**) The time series of the population density of brown planthoppers (B) tending to a vanishing level. (**c**) The time series of the population density of *Cyrtorhinus lividipennis* (C) exhibiting positive oscillation



(a) (b) (c)

Figure 2 A computer simulation of Eqs. (1a)–(1d). The solution of the system is permanent. Here, $a_1 = 0.5, a_2 = 0.7, b_1 = 0.5, r_1 = 0.9, d_1 = 0.01, d_2 = 0.1, h_1 = 3, h_2 = 5, h_3 = 0.2, \alpha = 0.2, \beta = 0.2, T = 10, B(0) = 5,$ and $C(0) = 5$. Here, all conditions in Theorem 2 are satisfied. (**a**) The solution trajectory projected on the (B, C)-plane. (**b**) The bounded time series of the population density of brown planthoppers (B). (**c**) The bounded time series of the population density of *Cyrtorhinus lividipennis* (C)



(a) (b) (c)

Figure 3 A computer simulation of Eqs. (1a)–(1d). The solution of the system is positive periodic. Here, $a_1 = 0.5, a_2 = 0.4, b_1 = 0.5, r_1 = 0.9, d_1 = 0.01, d_2 = 0.2, h_1 = 0.3, h_2 = 2, h_3 = 0.5, \alpha = 0.5, \beta = 0.5, T = 5, B(0) = 5,$ and $C(0) = 5$. Here, all conditions in Theorem 3 are satisfied. (**a**) The solution trajectory projected on (B, C)-plane. (**b**) The time series of the population density of brown planthoppers (B) and (**c**) The time series of population density of *Cyrtorhinus lividipennis* (C)

5 Conclusion

We investigate the dynamics behaviors of the populations of BPH and *Cyrtorhinus lividipennis* when insecticide is utilized to control the population of BPH in the paddy field through an impulsive mathematical model. Once insecticide is applied, both BPH and *Cyrtorhinus lividipennis* populations decrease rapidly. The appropriate duration T between two consecutive applications of insecticide might lead to effective control (the population of BPH reaches the vanishing level or maintains a level lower than the desired level) of BPH while *Cyrtorhinus lividipennis* still survive in the paddy field.

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The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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